M. Math. First Year
Complex Analysis
Supplementary Midsemestral Exam
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28th February 2024

Q 1. Find the image of the circle $|z| = R$ under the transformation $z \mapsto \frac{1}{2}(z + 1/z)$. Do this separately for $R > 1$ and $R = 1$.
Hint. The answers may be an ellipse or a line segment.

Q 2. Let u be a harmonic function on an open disc $D \subset \mathbb{C}$. Recall that this means that the 2nd order partial derivatives of u with respect to the variables x, y are continuous and $u_{xx} + u_{yy} = 0$. If u is non-constant, then prove that it cannot attain a maximum at a point of D.
Hint. Prove, and use the mean-value property.

OR

Q 2. Let f be an entire function satisfying $|f(z)| \leq log(1 + |z|)$ for all z. Prove that f must be a constant function.
Hint. Same idea as in Liouville's theorem, but applied to f' at a point.

Q 3. If f is a continuous function on $\mathbb{C}$ such that its restrictions to the subsets $\{z \in \mathbb{C} : x < 0\}$ and $\{z \in \mathbb{C} : x > 0\}$ are holomorphic, then show that f is entire.
Hint. Consider integrals over rectangles and, use Morera's theorem.

OR

Q 3. Let $f(z) = u(z) + iv(z)$ be holomorphic on the open unit disc D. If $u(0) = \pm v(0)$, prove that for all $0 < t < 1$, we have $\int_0^{2\pi} u(te^{i\theta})^2 d\theta = \int_0^{2\pi} v(te^{i\theta})^2 d\theta$.
Hint. Note $\Re f(0)^2 = u(0)^2 - v(0)^2$; and use Cauchy's formula for a suitable function on a suitable path.

Q 4. Prove that $\int_{C_r} \frac{dz}{z^2 \sin(z)} \to 0$ as $r \to \infty$, where C_r is the boundary of the square $|x|, |y| \leq (2r+1)\pi/2$. Using this, and the residue theorem, deduce that $\sum_{n\geq 1}(-1)^{n-1}/n^2 = \pi^2/12$.

OR

Q 4. Let f be holomorphic on a domain D, and satisfies $|f(z)^2 - 1| < 1$ for all $z \in D$. Show that $\Re(f)$ does not vanish on D, and has the same sign throughout D.

Q 5. Find a conformal map from $D \cap \{z : \Re(z) > 0\}$ onto D, where D is the open unit disc.
Hint. Map the domain onto the intersection of D with the upper half-plane; follow it up by mapping onto the first quadrant, and take the square map.

OR

Q 5. Let f be an entire function such that its restriction to $\mathbb{R}$ has real, non-negative values. Show that each REAL zero of f must have even order.

Q 6. For the Taylor series of the function $\frac{1}{1+z^2+z^4+z^6+z^8+z^{10}}$ around $z = 1$, determine the radius R of convergence.
Hint. R is the distance from 1 to the closest singularity.

OR

Q 6. If $p(z), q(z)$ are polynomials with $deg(p) < deg(q) + 1$, and if, $q(z)$ has simple zeroes at $z_1, \cdots, z_n$, show that for any large enough circle C, the integral $\int_C \frac{p(z)}{q(z)} dz = 0$. Deduce that $\sum_{i=1}^{n} p(z_i)/q'(z_i) = 0$.